Li, J., Liu, F., Feng, L., & Turner, I.
(2017)
A novel finite volume method for the Riesz space distributed-order advection–diffusion equation.
Applied Mathematical Modelling, *46*, pp. 536-553.

https://doi.org/10.1016/j.apm.2017.01.065

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A novel finite volume method for the Riesz space distributed-order advection-diffusion equation

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Abstract

In this paper, we investigate the finite volume method (FVM) for a distributed-order space-fractional advection-diffusion (AD) equation. The mid-point quadrature rule is used to approximate the distributed-order equation by a multi-term fractional model. Next, the transformed multi-term fractional equation is solved by discretizing in space by the finite volume method and in time using the Crank-Nicolson scheme. Especially, we use a novel technique to deal with the convection term, by which the Riesz fractional derivative of order $0 < \gamma < 1$ is transformed into a fractional integral form. And combining the nodal basis functions, we construct the discrete scheme, which is new and original. The unique solvability of the scheme is discussed. We also prove that the Crank-Nicolson scheme is unconditionally stable and convergent with second-order accuracy. Finally, we give some examples to show the effectiveness of the numerical method.

Keywords: Distributed-order equation; Finite volume method, Riesz fractional derivatives, Fractional advection-diffusion equation, Stability and convergence

1. Introduction

In the past few decades, there has been considerable interest in many areas such as natural sciences, biology, geological sciences, medicine, signal processing, etc. As they do not obey the Gaussian statistics, a host of scientists put more and more attentions on how to model them [17, 22, 24, 31, 37, 38, 39, 45]. In general, these models have forms of the single or multi-term time-, space-, or time-space-fractional differential equations. However, both the single and multi-term fractional equations are not suitable to simulate the diffusion processes in multi-fractal media which have no fixed scaling exponent, while distributed-order diffusion equations are shown to be useful tools to describe anomalous diffusion characterized by two or more scaling exponents in the mean squared displacement (MSD) or even by logarithmic time dependency of the MSD.

Caputo [5] first proposed the use of differential equations with distributed-order derivatives for generalizing stress-strain relations of unelastic media. Later, he [6, 7] discussed distributed-order time fractional differential equations and distributed-order space fractional differential equations, respectively and derived the solutions with closed form formulae of the classic problems. He found that one of the major differences between distributed-order time fractional differential equations and distributed-order space fractional differential equations is that the former represents the local variations and is particularly valid when considering local phenomena, while in an infinite medium it is more appropriate to introduce the space fractional order derivative to represent the effect of the medium and its space interaction with the fluid. Following on from this work, Chechkin, Sokolov et al. [9, 42] gave out diffusion-like equations with distributed-order time and space fractional derivatives for the kinetic description of anomalous diffusion and relaxation phenomena. They showed that the equations with the distributed-order derivatives on the proper side describe processes getting more anomalous in course of the time (accelerating superdiffusion and decelerating subdiffusion), while the equations with the additional distributed-order on the wrong side describe the situations getting less anomalous (decelerating superdiffusion and accelerating subdiffusion). In 2006, Meerschaert and Scheffler [34] developed a stochastic model based on random walks with a random waiting time between jumps. Scaling limits of these random walks were subordinated to random processes whose density functions solved the ultraslow diffusion equation. Umarov and Steinberg [46] constructed the multi-dimensional random walk models governed by distributed fractional order differential equations. In addition, they

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used the distributed-order differential equations to model the input-output relationship of linear time-variant system, some ultraslow and lateral diffusion processes, and to study of rheological properties of composite materials. Kochubei [25] also applied distributed-order diffusion equation to discuss ultraslow and lateral diffusion processes in 2008. In 2010, Caputo and Carcione [8] developed and solved a dissipative model for the propagation and attenuation of two-dimensional dilatational waves, using a new modeling algorithm based on distributed-order fractional time derivatives. Li et al. [26] applied the distributed-order filtering technique to model signal processing. In 2011, Atanackovic et al. [1] studied waves in a viscoelastic rod of finite length which was described by a constitutive equation of fractional distributed-order type with the special choice of weight functions. Eab and Lim [11] introduced the distributed-order fractional Langevin-like equations and applied them to describe anomalous diffusion without unique diffusion or scaling exponent. The distributed-order equations were also used to describe a variety of memory mechanisms and to represent the dispersion acting with several different relaxations (e.g. Anelastic relaxation mechanisms or spectral lines in the case of dielectric media) in [33]. In 2013, on the basis of a sub-diffusion model described by a distributed-order system of equations, Bulavatskya and Krivonosa [4] performed mathematical modeling of the dynamics of a locally nonequilibrium (in time) geomigration process in a geoporous medium saturated with a salt solution. Recently, Sandev et al. [40] studied distributed-order time fractional diffusion equations characterized by multifractal memory kernels, in contrast to the simple power-law kernel of common time fractional diffusion equations. Su et al. [43] presented a distributed-order fractional diffusion-wave equation (dofDWE) to describe radial groundwater flow to or from a well, and three sets of solutions of the dofDWE for aquifer tests: one for pumping tests, and two for slug tests, which were useful for gaining further insights into groundwater flow properties.

To date there are several papers focused on how to solve the distributed-order fractional equations. Meerschaert [35] investigated explicit strong solutions and stochastic analogues for time distributed-order fractional diffusion equations on bounded domains with Dirichlet boundary data. Gorenflo [16] and his co-workers provided the fundamental solution of the Cauchy problem for time distributed-order fractional equations by employing Laplace and Fourier transforms and interpreted the fundamental solution as a probability density. Luchko et al. [32] showed the uniqueness and continuous dependence on the initial data for the generalized distributed-order fractional diffusion equations on bounded domains. There are also a few papers that discuss the numerical solutions of distributed-order fractional equations. Ye et al. [30, 48, 49] applied implicit numerical method and compact difference scheme for time distributed-order fractional equations and obtained their convergence. Hu et al. [19, 20] used an implicit numerical method to discuss a time distributed-order two-sided space-fractional advection-dispersion equation and obtained stability and convergence criterion. Li et al. [27] applied reproducing kernel method to solve time distributed-order diffusion equations. Morgado et al. [36] put forward implicit scheme for time distributed-order reaction-diffusion equations with a nonlinear source term. Gao and Sun [14, 15, 44] focused on finite difference methods to solve one-dimensional and two-dimensional distributed-order differential equations and derived two alternating direction implicit difference schemes. They also used extrapolation method to improve the accuracy order and obtained high-order convergence rate. Wang and Liu [47] used shifted Grünwald–Letnikov to get the second-order accurate implicit numerical method for the Riesz space distributed-order advection-dispersion equations. Du, Hao and Sun [10] studied some high-order difference schemes for the distributed-order time-fractional equations in both one and two space dimensions. Based on the composite Simpson formula and Lubich second-order operator, they derived stable numerical solutions with higher order convergence rate in space.

However, to our best knowledge, there are only few works on the solution for distributed-order space fractional equations, especially their numerical solutions. In Section 5 of [42], Sokolov et al. discussed distributed-order space fractional diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \int_0^2 P(\alpha)\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha}d\alpha, \quad (\alpha \neq 1). \tag{1}$$

Where $P(\alpha)$ is a dimensional function of the order of the derivative α, $\frac{\partial^\alpha}{\partial|x|^\alpha}$ denotes the Riesz space functional derivative. In the general case $P(\alpha) = l^{\alpha-2}Kw(\alpha)$, l and K are dimensional positive constants, $[l] = cm, [K] = cm^2/sec$ and $w(\alpha) = A_1\delta(\alpha-\alpha_1) + A_2\delta(\alpha-\alpha_2)$, $0 < \alpha_1 < \alpha_2 \leq 2, A_1 > 0, A_2 > 0$. The equation for the characteristic function of Eq.(1) has the solution

$$g(k,t) = \exp -a_1|k|^{\alpha_1}t - a_2|k|^{\alpha_2}t \tag{2}$$

with $a_1 = A_1K/l^{2-\alpha_1}, a_2 = A_2K/l^{2-\alpha_2}$. Equation (2) is a product of two characteristic functions of Lévy stable PDFs with Lévy index α_1, α_2 and the scale parameters a_1^{1/α_1} and a_2^{1/α_2}, respectively. Through a series of analysis, Solokov et al. [42] concluded that at small times the characteristic displacement grew as t^{1/α_2}, whereas at large times it grew as t^{1/α_1}. It meant that the process was an accelerated superdiffusion.

Based on this model, in this paper, we dedicate to the numerical method for the following more general space distributed-order fractional advection-diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \lambda_1\int_1^2 P(\alpha)\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha}d\alpha + \lambda_2\int_0^1 Q(\gamma)\frac{\partial^\gamma u(x,t)}{\partial|x|^\gamma}d\gamma + f(x,t), \quad (x,t)\in(0,1)\times(0,1], \tag{3}$$

with boundary data

$$u(0,t)=0,\quad u(1,t)=0,\quad t\in(0,1] \tag{4}$$

and initial data

$$u(x,0)=\psi(x),\quad x\in[0,1]. \tag{5}$$

In Eq.(3), $\lambda_1>0,\lambda_2>0$, $P(\alpha)$, $Q(\gamma)$ are non-negative weight functions that satisfy the conditions

$$0<\int_1^2 P(\alpha)d\alpha<\infty,\ 0<\int_0^1 Q(\gamma)d\gamma<\infty,\ 1<\alpha<2,\ 0<\gamma<1.$$

We can see that when $\lambda_1=\lambda_2=1$, $P(\alpha)=Q(\gamma)$ and $f(x,t)=0$, Eq.(3) can be reduced to Eq.(1). Noting the advection orders are always close to 1, here we suppose that $Q(\gamma)$ vanish outside the interval $(1/2,1)$ [2, 3].

First, we use the mid-point quadrature rule to transform the space distributed-order diffusion Eq.(3) into a multi-term fractional equation [21, 28]. The treatment of the convection term is sparse, of which the existed methods all are finite difference methods [13, 41, 47] and there is no finite volume method or finite element method reported in the literature. As the order of the fractional derivative of the convection term is $0<\gamma<1$, it is difficult to utilise finite volume method to the convection term directly. Based on the definition of the fractional derivative, we rewrite the convection term as a kind of integral form, which is suitable to apply the finite volume method. Then combining the nodal basis functions, the discrete form of the convection term is obtained. It is worth to notice that the finite element method is also available to deal with the convection term with this technique, which is encouraging and promising. Therefore, it becomes the most significant contribution of this paper. And then by the finite volume method (FVM) [12, 18, 29], we solve the multi-term fractional equation and obtain the Crank-Nicolson scheme. Furthermore, we will prove that the Crank-Nicolson scheme is unconditionally stable and convergent with the accuracy of second order.

The structure of this paper is as follows. In section 2, we discretize the space distributed-order fractional equation into a multi-term fractional equation, then taking use of the finite volume method, we derive the Crank-Nicolson scheme for the transformed multi-term fractional equation. We prove the stability and convergence of the Crank-Nicolson iteration scheme in section 3. Finally, two examples are presented to show the effectiveness of our finite volume method in section 4 and some conclusions are drawn in section 5.

2. The Crank-Nicolson scheme with the finite volume method

We first introduce some preliminary definitions of the Riesz fractional derivatives. The Riesz fractional derivative is defined as follows: for $0<\alpha<2,\alpha\neq 1$,

$$\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha}=-\frac{1}{2\cos(\alpha\pi/2)}\left(\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}+\frac{\partial^\alpha u(x,t)}{\partial(-x)^\alpha}\right)$$

with

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\partial}{\partial x}\right)^n\int_0^x\frac{u(s,t)}{(x-s)^{\alpha-n+1}}ds,\quad n=[\alpha]+1,$$

$$\frac{\partial^\alpha u(x,t)}{\partial(-x)^\alpha}=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{\partial}{\partial x}\right)^n\int_x^1\frac{u(s,t)}{(s-x)^{\alpha-n+1}}ds,\quad n=[\alpha]+1,$$

and when $\alpha=n$ $(n=1,2)$, $\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha}=\frac{\partial^n u(x,t)}{\partial x^n}$. First, we discretize the integral interval $(1,2)$ in Eq.(3) for α by the grid $1=\xi_0<\xi_1<\cdots<\xi_S=2$ and denote $\Delta\xi_k=\xi_k-\xi_{k-1}=\frac{1}{S}=\sigma$, $k=1,2,\cdots,S$, $\alpha_k=\frac{\xi_k+\xi_{k-1}}{2}=1+\frac{2k-1}{2S}$. Also, discretize the integral interval $(0,1)$ for γ by the grid $0=\eta_0<\eta_1<\cdots<\eta_{\bar{S}}=1$ and denote $\Delta\eta_l=\eta_l-\eta_{l-1}=\frac{1}{\bar{S}}=\varrho$, $l=1,2,\cdots,\bar{S}$, $\gamma_l=\frac{\eta_l+\eta_{l-1}}{2}=\frac{2l-1}{2\bar{S}}$. Applying the mid-point quadrature rule, we obtain that

$$\int_1^2 P(\alpha)\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha}d\alpha=\sum_{k=1}^{S}P(\alpha_k)\frac{\partial^{\alpha_k}u(x,t)}{\partial|x|^{\alpha_k}}\Delta\xi_k+O(\sigma^2), \tag{6}$$

$$\int_0^1 Q(\gamma)\frac{\partial^\gamma u(x,t)}{\partial|x|^\gamma}d\gamma=\sum_{l=1}^{\bar{S}}Q(\gamma_l)\frac{\partial^{\gamma_l}u(x,t)}{\partial|x|^{\gamma_l}}\Delta\eta_l+O(\varrho^2). \tag{7}$$

Additionally, discretize the time domain $[0,1]$ by $t_n=n\tau, n=0,1,\cdots,N$ with $\tau=1/N$. Let S_h be a uniform partition of the space domain $[0,1]$, which is given by $x_i=ih, i=0,1,\cdots,M$ with $h=1/M$. Assume that $u(x,\cdot)\in C^3([0,1]), u(\cdot,t)\in C^2([0,1])$ and let $t_{n-\frac{1}{2}}=\frac{t_n+t_{n-1}}{2}$, then

$$\frac{\partial u(x,t_{n-\frac{1}{2}})}{\partial t}=\frac{u(x,t_n)-u(x,t_{n-1})}{\tau}+O(\tau^2). \tag{8}$$

Combining Eqs.(6)-(8), we obtain

$$\frac{u(x,t_n)-u(x,t_{n-1})}{\tau}=\frac{\lambda_1\sigma}{2}\sum_{k=1}^{S}P(\alpha_k)\left[\frac{\partial^{\alpha_k}u(x,t_n)}{\partial|x|^{\alpha_k}}+\frac{\partial^{\alpha_k}u(x,t_{n-1})}{\partial|x|^{\alpha_k}}\right]+\frac{\lambda_2\varrho}{2}\sum_{l=1}^{\bar{S}}Q(\gamma_l)\left[\frac{\partial^{\gamma_l}u(x,t_n)}{\partial|x|^{\gamma_l}}+\frac{\partial^{\gamma_l}u(x,t_{n-1})}{\partial|x|^{\gamma_l}}\right] \tag{9}$$

$$+\frac{1}{2}[f(x,t_n)+f(x,t_{n-1})]+O(\sigma^2+\varrho^2+\tau^2).$$

Note that

$$\frac{\partial^{\alpha_k}u(x,t)}{\partial|x|^{\alpha_k}}=-\frac{1}{2\cos\left(\frac{\alpha_k\pi}{2}\right)}\left[\frac{\partial^{\alpha_k}u(x,t)}{\partial x^{\alpha_k}}+\frac{\partial^{\alpha_k}u(x,t)}{\partial(-x)^{\alpha_k}}\right]=-\frac{1}{2\cos\left(\frac{\alpha_k\pi}{2}\right)}\frac{\partial}{\partial x}\left[\frac{\partial^{\beta_k}u(x,t)}{\partial x^{\beta_k}}-\frac{\partial^{\beta_k}u(x,t)}{\partial(-x)^{\beta_k}}\right], \tag{10}$$

with $\beta_k=\alpha_k-1$ and

$$\frac{\partial^{\gamma_l}u(x,t)}{\partial|x|^{\gamma_l}}=-\frac{1}{2\cos\left(\frac{\gamma_l\pi}{2}\right)}\left[\frac{\partial^{\gamma_l}u(x,t)}{\partial x^{\gamma_l}}+\frac{\partial^{\gamma_l}u(x,t)}{\partial(-x)^{\gamma_l}}\right]=-\frac{1}{2\cos\left(\frac{\gamma_l\pi}{2}\right)}\frac{\partial}{\partial x}\left[I_{0^+}^{1-\gamma_l}u(x,t)-I_{1^-}^{1-\gamma_l}u(x,t)\right], \tag{11}$$

where

$$I_{0^+}^{1-\gamma_l}u(x,t)=\frac{1}{\Gamma(1-\gamma_l)}\int_0^x\frac{u(\zeta,t)}{(x-\zeta)^{\gamma_l}}d\zeta,\quad I_{1^-}^{1-\gamma_l}u(x,t)=\frac{1}{\Gamma(1-\gamma_l)}\int_x^1\frac{u(\zeta,t)}{(\zeta-x)^{\gamma_l}}d\zeta.$$

Then, let $x_{i-\frac{1}{2}}=\frac{x_i+x_{i-1}}{2}, i=1,2,\cdots,M$ be the mid-point of the interval $[x_{i-1},x_i]$ and take the integration of the governing Eq.(9) over a control volume $[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}]$ for $i=1,2,\cdots,M-1$, which leads to

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}u(x,t_n)dx-\sum_{k=1}^{S}a_k\left[\frac{\partial^{\beta_k}u(x,t_n)}{\partial x^{\beta_k}}-\frac{\partial^{\beta_k}u(x,t_n)}{\partial(-x)^{\beta_k}}\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}-\sum_{l=1}^{\bar{S}}b_l\left[I_{0^+}^{1-\gamma_l}u(x,t_n)-I_{1^-}^{1-\gamma_l}u(x,t_n)\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tag{12}$$

$$=\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}u(x,t_{n-1})dx+\sum_{k=1}^{S}a_k\left[\frac{\partial^{\beta_k}u(x,t_{n-1})}{\partial x^{\beta_k}}-\frac{\partial^{\beta_k}u(x,t_{n-1})}{\partial(-x)^{\beta_k}}\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}+\sum_{l=1}^{\bar{S}}b_l\left[I_{0^+}^{1-\gamma_l}u(x,t_{n-1})-I_{1^-}^{1-\gamma_l}u(x,t_{n-1})\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$$

$$+\frac{\tau}{2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}[f(x,t_n)+f(x,t_{n-1})]dx+O(\tau h(\sigma^2+\varrho^2+\tau^2)),$$

where $a_k=-\frac{\lambda_1\sigma\tau P(\alpha_k)}{4\cos(\frac{\alpha_k\pi}{2})}>0, b_l=-\frac{\lambda_2\varrho\tau Q(\gamma_l)}{4\cos(\frac{\gamma_l\pi}{2})}\le 0$.

Now, we define the space V_h as the set of piecewise-linear polynomials on the mesh S_h. Then the approximate solution $u_h(x,t_n)\in P(0,1)$ with piecewise polynomials can be expressed as

$$u_h(x,t_n)=\sum_{j=1}^{M-1}u_j^n\phi_j(x) \tag{13}$$

with $\phi_i, 0\le i\le M$ being the nodal based functions of V_h. For more details, one can refer to [12]. Therefore, we obtain the subsequent Crank-Nicolson scheme:

$$\sum_{j=1}^{M-1}u_j^n\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\phi_j(x)dx-\sum_{j=1}^{M-1}u_j^n\sum_{k=1}^{S}a_k\left[\frac{d^{\beta_k}\phi_j(x)}{dx^{\beta_k}}-\frac{d^{\beta_k}\phi_j(x)}{d(-x)^{\beta_k}}\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}-\sum_{j=1}^{M-1}u_j^n\sum_{l=1}^{\bar{S}}b_l\left[I_{0^+}^{1-\gamma_l}\phi_j(x)-I_{1^-}^{1-\gamma_l}\phi_j(x)\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$$

$$=\sum_{j=1}^{M-1}u_j^{n-1}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\phi_j(x)dx+\sum_{j=1}^{M-1}u_j^{n-1}\sum_{k=1}^{S}a_k\left[\frac{d^{\beta_k}\phi_j(x)}{dx^{\beta_k}}-\frac{d^{\beta_k}\phi_j(x))}{d(-x)^{\beta_k}}\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}+\sum_{j=1}^{M-1}u_j^{n-1}\sum_{l=1}^{\bar{S}}b_l\left[I_{0^+}^{1-\gamma_l}\phi_j(x)-I_{1^-}^{1-\gamma_l}\phi_j(x)\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$$

$$+\frac{\tau}{2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}[f(x,t_n)+f(x,t_{n-1})]dx.$$

By direct calculations, it follows that for $1\le i,j\le M-1$,

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\phi_j(x)dx=\begin{cases}h/8, & |i-j|=1,\\ 3h/4, & i=j,\\ 0, & \text{else.}\end{cases}\qquad \text{and}$$

$$\frac{d^{\beta_k}\phi_j(x_{i+\frac{1}{2}})}{dx^{\beta_k}} = \frac{1}{\Gamma(2-\beta_k)h^{\beta_k}}\begin{cases} 0, & j>i+1,\\ 2^{\beta_k-1}, & j=i+1,\\ (3/2)^{1-\beta_k}-2^{\beta_k}, & j=i,\\ c^k_{i-j+1}, & j<i,\end{cases} \quad \frac{d^{\beta_k}\phi_j(x_{i-\frac{1}{2}})}{dx^{\beta_k}} = \frac{1}{\Gamma(2-\beta_k)h^{\beta_k}}\begin{cases} 0, & j>i,\\ 2^{\beta_k-1}, & j=i,\\ (3/2)^{1-\beta_k}-2^{\beta_k}, & j=i-1,\\ c^k_{i-j}, & j<i-1,\end{cases}$$

$$\frac{d^{\beta_k}\phi_j(x_{i+\frac{1}{2}})}{d(-x)^{\beta_k}} = \frac{1}{\Gamma(2-\beta_k)h^{\beta_k}}\begin{cases} c^k_{j-i}, & j>i+1,\\ (3/2)^{1-\beta_k}-2^{\beta_k}, & j=i+1,\\ 2^{\beta_k-1}, & j=i,\\ 0, & j<i,\end{cases} \quad \frac{d^{\beta_k}\phi_j(x_{i-\frac{1}{2}})}{d(-x)^{\beta_k}} = \frac{1}{\Gamma(2-\beta_k)h^{\beta_k}}\begin{cases} c^k_{j-i+1}, & j>i,\\ (3/2)^{1-\beta_k}-2^{\beta_k}, & j=i,\\ 2^{\beta_k-1}, & j=i-1,\\ 0, & j<i-1,\end{cases}$$

here, $c_i^k = (i-\frac{3}{2})^{1-\beta_k} - 2(i-\frac{1}{2})^{1-\beta_k} + (i+\frac{1}{2})^{1-\beta_k}, i = 2,3,\cdots$ In addition,

$$I_{0^+}^{1-\gamma_l}\phi_j(x_{i+\frac{1}{2}}) = \frac{h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}\begin{cases} 0, & j>i+1,\\ (1/2)^{2-\gamma_l}, & j=i+1,\\ (3/2)^{2-\gamma_l}-2(1/2)^{2-\gamma_l}, & j=i,\\ d^l_{i-j+1}, & j<i,\end{cases}$$

$$I_{0^+}^{1-\gamma_l}\phi_j(x_{i-\frac{1}{2}}) = \frac{h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}\begin{cases} 0, & j>i,\\ (1/2)^{2-\gamma_l}, & j=i,\\ (3/2)^{2-\gamma_l}-2(1/2)^{2-\gamma_l}, & j=i-1,\\ d^l_{i-j}, & j<i-1,\end{cases}$$

$$I_{1^-}^{1-\gamma_l}\phi_j(x_{i+\frac{1}{2}}) = \frac{h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}\begin{cases} d^l_{j-i}, & j>i+1,\\ (3/2)^{2-\gamma_l}-2(1/2)^{2-\gamma_l}, & j=i+1,\\ (1/2)^{2-\gamma_l}, & j=i,\\ 0, & j<i,\end{cases}$$

$$I_{1^-}^{1-\gamma_l}\phi_j(x_{i-\frac{1}{2}}) = \frac{h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}\begin{cases} d^l_{j-i+1}, & j>i,\\ (3/2)^{2-\gamma_l}-2(1/2)^{2-\gamma_l}, & j=i,\\ (1/2)^{2-\gamma_l}, & j=i-1,\\ 0, & j<i-1\end{cases}$$

with $d_i^l = (i-\frac{3}{2})^{2-\gamma_l} - 2(i-\frac{1}{2})^{2-\gamma_l} + (i+\frac{1}{2})^{2-\gamma_l}, i = 2,3,\cdots$. Henceforth,

$$\begin{aligned}\frac{h}{8}(u_{i-1}^n + 6u_i^n + u_{i+1}^n) - \sum_{j=1}^{M-1} u_j^n G_{ij} - \sum_{j=1}^{M-1} u_j^n D_{ij} = \frac{h}{8}(u_{i-1}^{n-1} + 6u_i^{n-1}\\ +u_{i+1}^{n-1}) + \sum_{j=1}^{M-1} u_j^{n-1} G_{ij} + \sum_{j=1}^{M-1} u_j^{n-1} D_{ij} + \frac{\tau}{2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [f(x,t_n) + f(x,t_{n-1})]dx.\end{aligned} \tag{14}$$

where $G_{ij} = G_{1,ij} - G_{2,ij}, D_{ij} = D_{1,ij} - D_{2,ij}$ and

$$G_{1,ij} = \sum_{k=1}^{S} a_k[\frac{d^{\beta_k}\phi_j(x_{i+\frac{1}{2}})}{dx^{\beta_k}} - \frac{d^{\beta_k}\phi_j(x_{i-\frac{1}{2}})}{dx^{\beta_k}}], \qquad G_{2,ij} = \sum_{k=1}^{S} a_k[\frac{d^{\beta_k}\phi_j(x_{i+\frac{1}{2}})}{d(-x)^{\beta_k}} - \frac{d^{\beta_k}\phi_j(x_{i-\frac{1}{2}})}{d(-x)^{\beta_k}}],$$

$$D_{1,ij} = \sum_{l=1}^{\bar{S}} b_l[I_{0^+}^{1-\gamma_l}\phi_j(x_{i+\frac{1}{2}}) - I_{0^+}^{1-\gamma_l}\phi_j(x_{i-\frac{1}{2}})], \qquad D_{2,ij} = \sum_{l=1}^{\bar{S}} b_l[I_{1^-}^{1-\gamma_l}\phi_j(x_{i+\frac{1}{2}}) - I_{1^-}^{1-\gamma_l}\phi_j(x_{i-\frac{1}{2}})].$$

For $i = 1,2,\cdots,M-1$, denote $(F^n)_i = \frac{\tau}{2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}[f(x,t_n) + f(x,t_{n-1})]dx$, $A_{ij} = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\phi_j(x)dx$, $U^n = (u_1^n, u_2^n, \cdots, u_{M-1}^n)^T$. Then, Eq.(14) can be expressed in matrix form as

$$(A - G - D)U^n = (A + G + D)U^{n-1} + F^n. \tag{15}$$

The initial condition is discretized as $\psi_i = \psi(ih)$ for $i = 0,1,2,\cdots,M$ and $U^0 = (u_1^0, u_2^0, \cdots, u_{M-1}^0)^T = (\psi(h), \psi(2h), \cdots, \psi((M-1)h))^T$.

3. Stability and convergence of Crank-Nicolson scheme

Along the same lines as the proof of Lemma 1 in [12], we can obtain the following two lemmas.

Lemma 1. *For* $0 < \beta_k = \alpha_k - 1 < 1$, $c_i^k = (i-\frac{3}{2})^{1-\beta_k} - 2(i-\frac{1}{2})^{1-\beta_k} + (i+\frac{1}{2})^{1-\beta_k}$, $i = 2, 3, \cdots$, *the following hold:*

(1) c_i^k *is increasing monotonically as i increases, and* $c_i^k < 0, i = 2, 3, \cdots$;

(2) $\lim\limits_{i\to+\infty} c_i^k = 0$;

(3) $\sum\limits_{i=2}^{+\infty}(c_{i+1}^k - c_i^k) = -c_2^k$.

Lemma 2. *For* $0 < \gamma_l < 1$, $d_i^l = (i-\frac{3}{2})^{2-\gamma_l} - 2(i-\frac{1}{2})^{2-\gamma_l} + (i+\frac{1}{2})^{2-\gamma_l}$, $i = 2, 3, \cdots$, *the following hold:*

(1) d_i^l *is decreasing monotonically as i increases, and* $d_i^l > 0, i = 2, 3, \cdots$;

(2) $\lim\limits_{i\to+\infty} d_i^l = 0$;

(3) $\sum\limits_{i=2}^{+\infty}(d_{i+1}^l - d_i^l) = -d_2^l$.

Also, following a proof similar to Theorem 1 in [12], we can obtain:

Theorem 1. *For* $0 < \beta_k < 1$, *the coefficients* G_{ij} *satisfy*

$$|G_{ii}| > \sum_{j=1, j\neq i}^{M-1} |G_{ij}|, \ i = 1, 2, \cdots, M-1,$$

i.e., G is strictly diagonally dominant.

Theorem 2. *For* $0 < \beta_k < 1$, $\frac{1}{2} < \gamma_l < 1$, $B = A - G - D$, *then B is also strictly diagonally dominant and the spectral radius of* B^{-1} *fulfills*

$$\rho(B^{-1}) < \frac{2}{h}. \tag{16}$$

Proof.

$$B_{ij} = \begin{cases} \sum\limits_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}(c_{j-i}^k - c_{j-i+1}^k) + \sum\limits_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}(d_{j-i}^l - d_{j-i+1}^l), & j > i+1, \\ h/8 - \sum\limits_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[3(\frac{1}{2})^{1-\beta_k} - (\frac{3}{2})^{1-\beta_k} + c_2^k] & \\ - \sum\limits_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[3(\frac{1}{2})^{2-\gamma_l} - (\frac{3}{2})^{2-\gamma_l} + d_2^l], & j = i+1, \\ 3h/4 - \sum\limits_{k=1}^{S} \frac{2a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[(\frac{3}{2})^{1-\beta_k} - 3(\frac{1}{2})^{1-\beta_k}] & \\ - \sum\limits_{l=1}^{\bar{S}} \frac{2b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[(\frac{3}{2})^{2-\gamma_l} - 3(\frac{1}{2})^{2-\gamma_l}], & j = i, \\ h/8 - \sum\limits_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[3(\frac{1}{2})^{1-\beta_k} - (\frac{3}{2})^{1-\beta_k} + c_2^k] & \\ - \sum\limits_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[3(\frac{1}{2})^{2-\gamma_l} - (\frac{3}{2})^{2-\gamma_l} + d_2^l], & j = i-1, \\ \sum\limits_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}(c_{i-j}^k - c_{i-j+1}^k) + \sum\limits_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}(d_{i-j}^l - d_{i-j+1}^l), & j < i-1. \end{cases}$$

Since $a_k > 0, b_l < 0$, from Lemma 1 and Lemma 2, we know $B_{ij} < 0$ for $j > i+1$ and $j < i-1$. Note that

$$\left(\frac{3}{2}\right)^{1-\beta_k} - 3\left(\frac{1}{2}\right)^{1-\beta_k} = \left(\frac{1}{2}\right)^{1-\beta_k}(3^{1-\beta_k} - 3) < 0, \tag{17}$$

$$\left(\frac{3}{2}\right)^{2-\gamma_l} - 3\left(\frac{1}{2}\right)^{2-\gamma_l} = \left(\frac{1}{2}\right)^{2-\gamma_l}(3^{2-\gamma_l} - 3) > 0, \tag{18}$$

which asserts $B_{ii}>0$. In addition,

$$3\left(\frac{1}{2}\right)^{1-\beta_k}-\left(\frac{3}{2}\right)^{1-\beta_k}+c_2^k=\left(\frac{1}{2}\right)^{1-\beta_k}(4-3\cdot 3^{1-\beta_k}+5^{1-\beta_k}), \tag{19}$$

$$3\left(\frac{1}{2}\right)^{2-\gamma_l}-\left(\frac{3}{2}\right)^{2-\gamma_l}+d_2^l=\left(\frac{1}{2}\right)^{2-\gamma_l}(4-3\cdot 3^{2-\gamma_l}+5^{2-\gamma_l}). \tag{20}$$

Let $g(x)=4-3\cdot 3^x+5^x$, we know that $g(x)>0$ for $x\in(0,1)$ and $g(x)<0$ for $x\in(1,3/2)$. Thus, for $0<\beta_k<1$ and $\frac{1}{2}<\gamma_l<1$, we have $3\cdot(\frac{1}{2})^{1-\beta_k}-(\frac{3}{2})^{1-\beta_k}+c_2^k>0$ and $3\cdot(\frac{1}{2})^{2-\gamma_l}-(\frac{3}{2})^{2-\gamma_l}+d_2^l<0$. Hence

$$\begin{aligned}
&\sum_{j=1,j\neq i}^{M-1}|B_{ij}|=\sum_{j=1}^{i-2}|B_{ij}|+\sum_{j=i+2}^{M-1}|B_{ij}|+|B_{i,i-1}|+|B_{i,i+1}|\\
&<\sum_{k=1}^{S}\frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}\sum_{j=-\infty}^{i-2}(c_{i-j+1}^k-c_{i-j}^k)+\sum_{l=1}^{\bar{S}}\frac{b_lh^{1-\gamma_l}}{\Gamma(3-\gamma_l)}\sum_{j=-\infty}^{i-2}(d_{i-j+1}^l-d_{i-j}^l)+\sum_{k=1}^{S}\frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}\sum_{j=i+2}^{+\infty}(c_{j-i+1}^k-c_{j-i}^k)\\
&\quad+\sum_{l=1}^{\bar{S}}\frac{b_lh^{1-\gamma_l}}{\Gamma(3-\gamma_l)}\sum_{j=i+2}^{+\infty}(d_{j-i+1}^l-d_{j-i}^l)+\frac{h}{4}+\sum_{k=1}^{S}\frac{2a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[3(\frac{1}{2})^{1-\beta_k}-(\frac{3}{2})^{1-\beta_k}+c_2^k]+\sum_{l=1}^{\bar{S}}\frac{2b_lh^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[3(\frac{1}{2})^{2-\gamma_l}-(\frac{3}{2})^{2-\gamma_l}+d_2^l]\\
&=\sum_{k=1}^{S}\frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}\cdot(-2c_2^k)+\sum_{l=1}^{\bar{S}}\frac{b_lh^{1-\gamma_l}}{\Gamma(3-\gamma_l)}\cdot(-2d_2^l)+\frac{h}{4}+\sum_{k=1}^{S}\frac{2a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[3(\frac{1}{2})^{1-\beta_k}-(\frac{3}{2})^{1-\beta_k}+c_2^k]\\
&\quad+\sum_{l=1}^{\bar{S}}\frac{2b_lh^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[3(\frac{1}{2})^{2-\gamma_l}-(\frac{3}{2})^{2-\gamma_l}+d_2^l]\\
&=\frac{h}{4}+\sum_{k=1}^{S}\frac{2a_k[3(\frac{1}{2})^{1-\beta_k}-(\frac{3}{2})^{1-\beta_k}]}{\Gamma(2-\beta_k)h^{\beta_k}}+\sum_{l=1}^{\bar{S}}\frac{2b_lh^{1-\gamma_l}[3(\frac{1}{2})^{2-\gamma_l}-(\frac{3}{2})^{2-\gamma_l}]}{\Gamma(3-\gamma_l)}\\
&<\frac{3h}{4}+\sum_{k=1}^{S}\frac{2a_k[3(\frac{1}{2})^{1-\beta_k}-(\frac{3}{2})^{1-\beta_k}]}{\Gamma(2-\beta_k)h^{\beta_k}}+\sum_{l=1}^{\bar{S}}\frac{2b_lh^{1-\gamma_l}[3(\frac{1}{2})^{2-\gamma_l}-(\frac{3}{2})^{2-\gamma_l}]}{\Gamma(3-\gamma_l)}\\
&=B_{ii}.
\end{aligned}$$

The proof of Eq.(16) follows **part b** of Theorem 2 in [12]. □

Since $B=A-G-D$ is strictly diagonally dominant, then B is nonsingular and invertible. The Crank-Nicolson scheme can be rewritten as

$$U^n=B^{-1}(A+G+D)U^{n-1}+B^{-1}F. \tag{21}$$

Theorem 3. *Define $W=(\lambda-1)A-(\lambda+1)(G+D)$, for $0<\beta_k<1$, $\frac{1}{2}<\gamma_l<1$, if $\lambda>1$ or $\lambda<-1$, we can conclude that W is strictly diagonally dominant, i.e.,*

$$|W_{ii}|>\sum_{j=1,j\neq i}^{M-1}|W_{ij}|,\ i=1,2,\cdots,M-1.$$

Proof.

$$W_{ij}=\begin{cases}
(\lambda+1)[\sum_{k=1}^{S}\frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}(c_{j-i}^k-c_{j-i+1}^k)+\sum_{l=1}^{\bar{S}}\frac{b_lh^{1-\gamma_l}}{\Gamma(3-\gamma_l)}(d_{j-i}^l-d_{j-i+1}^l)], & j>i+1,\\
(\lambda-1)\frac{h}{8}-(\lambda+1)\{\sum_{k=1}^{S}\frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[3(\frac{1}{2})^{1-\beta_k}-(\frac{3}{2})^{1-\beta_k}+c_2^k] & \\
+\sum_{l=1}^{\bar{S}}\frac{b_lh^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[3(\frac{1}{2})^{2-\gamma_l}-(\frac{3}{2})^{2-\gamma_l}+d_2^l]\}, & j=i+1,\\
(\lambda-1)\frac{3h}{4}-(\lambda+1)\{\sum_{k=1}^{S}\frac{2a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[(\frac{3}{2})^{1-\beta_k}-3(\frac{1}{2})^{1-\beta_k}] & \\
+\sum_{l=1}^{\bar{S}}\frac{2b_lh^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[(\frac{3}{2})^{2-\gamma_l}-3(\frac{1}{2})^{2-\gamma_l}]\}, & j=i,\\
(\lambda-1)\frac{h}{8}-(\lambda+1)\{\sum_{k=1}^{S}\frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[3(\frac{1}{2})^{1-\beta_k}-(\frac{3}{2})^{1-\beta_k}+c_2^k] & \\
+\sum_{l=1}^{\bar{S}}\frac{b_lh^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[3(\frac{1}{2})^{2-\gamma_l}-(\frac{3}{2})^{2-\gamma_l}+d_2^l]\}, & j=i-1,\\
(\lambda+1)[\sum_{k=1}^{S}\frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}(c_{i-j}^k-c_{i-j+1}^k)+\sum_{l=1}^{\bar{S}}\frac{b_lh^{1-\gamma_l}}{\Gamma(3-\gamma_l)}(d_{i-j}^l-d_{i-j+1}^l)], & j<i-1.
\end{cases}$$

(1) If $\lambda > 1$, since $a_k > 0, b_l < 0$, (17) and (18) yield that $W_{ii} > 0$. From Lemma 1 and Lemma 2, it is obvious that $W_{ii} > 0$ and $W_{ij} < 0$ for $j > i+1$ and $j < i-1$. Now, we focus on the sign of $W_{i,i-1}$ and $W_{i,i+1}$.

(i) If $W_{i,i-1} = W_{i,i+1} = (\lambda-1)\frac{h}{8} - (\lambda+1)\{\sum_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[3(\frac{1}{2})^{1-\beta_k} - (\frac{3}{2})^{1-\beta_k} + c_2^k] + \sum_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[3(\frac{1}{2})^{2-\gamma_l} - (\frac{3}{2})^{2-\gamma_l} + d_2^l]\} < 0$, then

$$
\begin{aligned}
\sum_{j=1,j\neq i}^{M-1} |W_{ij}| &= \sum_{j=1}^{i-2} |W_{ij}| + \sum_{j=i+2}^{M-1} |W_{ij}| + |W_{i,i-1}| + |W_{i,i+1}| \\
&< (\lambda+1)[\sum_{j=-\infty}^{i-2}\sum_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}(c_{i-j+1}^k - c_{i-j}^k) + \sum_{j=-\infty}^{i-2}\sum_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}(d_{i-j+1}^l - d_{i-j}^l) + \sum_{j=i+2}^{+\infty}\sum_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}(c_{j-i+1}^k - c_{j-i}^k) \\
&\quad + \sum_{j=i+2}^{+\infty}\sum_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}(d_{j-i+1}^l - d_{j-i}^l)] - (\lambda-1)\frac{h}{4} + 2(\lambda+1)\{\sum_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[3(\frac{1}{2})^{1-\beta_k} - (\frac{3}{2})^{1-\beta_k} + c_2^k] \\
&\quad + \sum_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[3(\frac{1}{2})^{2-\gamma_l} - (\frac{3}{2})^{2-\gamma_l} + d_2^l]\} \\
&< (\lambda-1)\frac{3h}{4} + 2(\lambda+1)\{\sum_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[3(\frac{1}{2})^{1-\beta_k} - (\frac{3}{2})^{1-\beta_k}] + \sum_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[3(\frac{1}{2})^{2-\gamma_l} - (\frac{3}{2})^{2-\gamma_l}]\} = W_{ii}.
\end{aligned}
$$

Therefore,

$$
|W_{ii}| > \sum_{j=1,j\neq i}^{M-1} |W_{ij}|.
$$

(ii) If $W_{i,i-1} = W_{i,i+1} = (\lambda-1)\frac{h}{8} - (\lambda+1)\{\sum_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[3(\frac{1}{2})^{1-\beta_k} - (\frac{3}{2})^{1-\beta_k} + c_2^k] + \sum_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[3(\frac{1}{2})^{2-\gamma_l} - (\frac{3}{2})^{2-\gamma_l} + d_2^l]\} \geq 0$, then

$$
\begin{aligned}
\sum_{j=1,j\neq i}^{M-1} |W_{ij}| &= \sum_{j=1}^{i-2} |W_{ij}| + \sum_{j=i+2}^{M-1} |W_{ij}| + |W_{i,i-1}| + |W_{i,i+1}| \\
&< (\lambda+1)[\sum_{j=-\infty}^{i-2}\sum_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}(c_{i-j+1}^k - c_{i-j}^k) + \sum_{j=-\infty}^{i-2}\sum_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}(d_{i-j+1}^l - d_{i-j}^l) + \sum_{j=i+2}^{+\infty}\sum_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}(c_{j-i+1}^k - c_{j-i}^k) \\
&\quad + \sum_{j=i+2}^{+\infty}\sum_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}(d_{j-i+1}^l - d_{j-i}^l)] + (\lambda-1)\frac{h}{4} - 2(\lambda+1)\{\sum_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[3(\frac{1}{2})^{1-\beta_k} - (\frac{3}{2})^{1-\beta_k} + c_2^k] \\
&\quad + \sum_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[3(\frac{1}{2})^{2-\gamma_l} - (\frac{3}{2})^{2-\gamma_l} + d_2^l]\} \\
&= (\lambda-1)\frac{h}{4} - 2(\lambda+1)\{\sum_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[5(\frac{1}{2})^{1-\beta_k} - 5(\frac{3}{2})^{1-\beta_k} + 2(\frac{5}{2})^{1-\beta_k}] \\
&\quad + \sum_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[5(\frac{1}{2})^{2-\gamma_l} - 5(\frac{3}{2})^{2-\gamma_l} + 2(\frac{5}{2})^{2-\gamma_l}]\} \\
&< (\lambda-1)\frac{3h}{4} + 2(\lambda+1)\{\sum_{k=1}^{S} \frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[3(\frac{1}{2})^{1-\beta_k} - (\frac{3}{2})^{1-\beta_k}] + \sum_{l=1}^{\bar{S}} \frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[3(\frac{1}{2})^{2-\gamma_l} - (\frac{3}{2})^{2-\gamma_l}]\} = W_{ii}
\end{aligned}
$$

as

$$2(\lambda+1)\{\sum_{k=1}^{S}\frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}[8(\frac{1}{2})^{1-\beta_k}-6(\frac{3}{2})^{1-\beta_k}+2(\frac{5}{2})^{1-\beta_k}]+\sum_{l=1}^{\bar{S}}\frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}[8(\frac{1}{2})^{2-\gamma_l}-6(\frac{3}{2})^{2-\gamma_l}+2(\frac{5}{2})^{2-\gamma_l}]\}$$
$$=4(\lambda+1)\{\sum_{k=1}^{S}\frac{a_k}{\Gamma(2-\beta_k)h^{\beta_k}}(\frac{1}{2})^{1-\beta_k}[4-3\cdot 3^{1-\beta_k}+5^{1-\beta_k}]+\sum_{l=1}^{\bar{S}}\frac{b_l h^{1-\gamma_l}}{\Gamma(3-\gamma_l)}(\frac{1}{2})^{2-\gamma_l}[4-3\cdot 3^{2-\gamma_l}+5^{2-\gamma_l}]\}$$
$$>0>-(\lambda-1)\frac{h}{2}.$$

Henceforth, $|W_{ii}|>\sum_{j=1,j\neq i}^{M-1}|W_{ij}|$.

(2) If $\lambda<-1$, it can be easily seen that $W_{ii}<0$ and $W_{ij}>0$ for $j>i+1$ and $j<i-1$. Along the same line of the proof for the case $\lambda>1$, we also can obtain $|W_{ii}|>\sum_{j=1,j\neq i}^{M-1}|W_{ij}|$.

This proof is completed. □

Theorem 4. *The spectral radius of $B^{-1}(A+G+D)$ satisfies $\rho(B^{-1}(A+G+D))<1$, hence the Crank-Nicolson scheme (21) is unconditionally stable.*

Proof. Since B, A, G and D are symmetric positive definite, it is easy to conclude that B^{-1} is symmetric positive definite and

$$B^{-1}(A+G+D)=B^{-\frac{1}{2}}(B^{-\frac{1}{2}}(A+G+D)B^{-\frac{1}{2}})B^{\frac{1}{2}},$$

which means that $B^{-1}(A+G+D)\sim B^{-\frac{1}{2}}(A+G+D)B^{-\frac{1}{2}}$. Thus, $B^{-1}(A+G+D)$ and $B^{-\frac{1}{2}}(A+G+D)B^{-\frac{1}{2}}$ have the same eigenvalues. The symmetric positive definiteness of A, G, D and B gives that $B^{-\frac{1}{2}}(A+G+D)B^{-\frac{1}{2}}$ is symmetric positive definite. Thus, all the eigenvalues of $B^{-\frac{1}{2}}(A+G+D)B^{-\frac{1}{2}}$ are real and so are $B^{-1}(A+G+D)$. Suppose λ is an eigenvalue of $B^{-1}(A+G+D)$, and

$$\det(\lambda I-B^{-1}(A+G+D))=\det(B^{-1})\cdot\det(\lambda B-(A+G+D))=0. \tag{22}$$

Note that B is nonsingular and invertible, then $\det(B^{-1})\neq 0$, thus $\det(\lambda B-(A+G+D))=0$. Let $W=\lambda B-(A+G+D)$, then

$$W=(\lambda-1)A-(\lambda+1)(G+D)$$

(i) If $\lambda=\pm 1$ or 0, it is obvious that W is diagonally dominant;

(ii) If $\lambda>1$ or $\lambda<-1$, Theorem 3 asserts that W is also diagonally dominant.

Therefore, $\det(W)\neq 0$ for all $\lambda\geq 1$ or $\lambda\leq -1$ or $\lambda=0$. According to the above analysis, as the roots of the equation $\det(\lambda B-(A+G+D))=0$ exist, λ must satisfy $-1<\lambda<0$ or $0<\lambda<1$, which means the eigenvalue of $B^{-1}(A+G+D)$ satisfy $|\lambda|<1$. Thus, $\rho(B^{-1}(A+G+D))<1$, which completes the proof. □

By following the proof in Corollary 1 [12], it follows that

Lemma 3. *For $0<\beta_k<1$, $\frac{1}{2}<\gamma_l<1$, if $\frac{\partial^{\beta_k+1}u(x,t)}{\partial x^{\beta_k+1}}\in L^1(\mathbb{R})$, then*

$$\frac{\partial^{\beta_k}u(x,t_n)}{\partial x^{\beta_k}}\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}=\sum_{j=1}^{M-1}u_j^n\left(\frac{d^{\beta_k}\phi_j(x_{i+\frac{1}{2}})}{dx^{\beta_k}}-\frac{d^{\beta_k}\phi_j(x_{i-\frac{1}{2}})}{dx^{\beta_k}}\right)+O(h^3), \tag{23}$$

$$\frac{\partial^{\beta_k}u(x,t_n)}{\partial(-x)^{\beta_k}}\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}=\sum_{j=1}^{M-1}u_j^n\left(\frac{d^{\beta_k}\phi_j(x_{i+\frac{1}{2}})}{d(-x)^{\beta_k}}-\frac{d^{\beta_k}\phi_j(x_{i-\frac{1}{2}})}{d(-x)^{\beta_k}}\right)+O(h^3), \tag{24}$$

$$I_{0^+}^{1-\gamma_l}u(x,t_n)\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}=\sum_{j=1}^{M-1}u_j^n(I_{0^+}^{1-\gamma_l}\phi_j(x_{i+\frac{1}{2}})-I_{0^+}^{1-\gamma_l}\phi_j(x_{i-\frac{1}{2}}))+O(h^3), \tag{25}$$

$$I_{1^-}^{1-\gamma_l}u(x,t_n)\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}=\sum_{j=1}^{M-1}u_j^n\left(I_{1^-}^{1-\gamma_l}\phi_j(x_{i+\frac{1}{2}})-I_{1^-}^{1-\gamma_l}\phi_j(x_{i-\frac{1}{2}})\right)+O(h^3). \tag{26}$$

Theorem 5. *Let u^n be the exact solution of the problem (3). Then the numerical solution U^n unconditionally converges to the exact solution u^n as h, τ and σ, ϱ tend to zero. Moreover,*

$$\|u^n - U^n\| \le C(\sigma^2 + \varrho^2 + \tau^2 + h^2),$$

where $u^n = (u(x_1, t_n), u(x_2, t_n), \cdots, u(x_{M-1}, t_n))$, $U^n = (u_1^n, u_2^n, \cdots, u_{M-1}^n)$.

Proof. Let e_i^n denote the error at the point (x_i, t_n). Substituting $e_i^n = u(x_i, t_n) - u_i^n$ into (14) and combining (12) with Eqs.(23)-(26) yields that

$$\frac{h}{8}(e_{i-1}^n + 6e_i^n + e_{i+1}^n) - \sum_{j=1}^{M-1} e_j^n G_{ij} - \sum_{j=1}^{M-1} e_j^n D_{ij} = \frac{h}{8}(e_{i-1}^{n-1} + 6e_i^{n-1} + e_{i+1}^{n-1})$$
$$+ \sum_{j=1}^{M-1} e_j^{n-1} G_{ij} + \sum_{j=1}^{M-1} e_j^{n-1} D_{ij} + O(\tau h(\sigma^2 + \varrho^2 + \tau^2 + h^2)).$$

Note that $e_0^n = e_M^n = 0$ and $e_i^0 = 0$ for $i = 1, 2, \cdots, M-1$. Thus,

$$BE^n = (A + G + D)E^{n-1} + O(\tau h(\sigma^2 + \varrho^2 + \tau^2 + h^2))\chi,$$

here $\chi = (1, 1, \cdots, 1)^T, E^n = (e_1^n, e_2^n, \cdots, e_{M-1}^n)^T$. Setting $Q = B^{-1}(A + G + D)$ and $\mathbf{b} = O(\tau h(\sigma^2 + \varrho^2 + \tau^2 + h^2))B^{-1}$, by iteration, one has

$$E^n = (Q^{n-1} + Q^{n-2} + \cdots + I)\mathbf{b}.$$

According to Theorem 2, we know that $\rho(B^{-1}) < \frac{2}{h}$ and $\rho(Q) < 1$, then there exists a vector norm and induced matrix norm $\|\cdot\|$ such that $\|Q\| < 1$ and $\|B^{-1}\| < Ch^{-1}$. Then upon taking norms, we have

$$\|E^n\| \le (\|Q^{n-1}\| + \|Q^{n-2}\| + \cdots + 1)\|\mathbf{b}\| \le n\|\mathbf{b}\| \le O(\sigma^2 + \varrho^2 + \tau^2 + h^2).$$

Therefore,
$$\|E^n\| \le C(\sigma^2 + \varrho^2 + \tau^2 + h^2). \qquad \square$$

4. Numerical examples

In order to illustrate the behavior of our numerical method and demonstrate the effectiveness of our theoretical analysis, some examples are given.

4.1. Example 1

Firstly, we consider the following distributed-order equation [47]:

$$\frac{\partial u(x,t)}{\partial t} = \int_1^2 P(\alpha)\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} d\alpha + \int_0^1 Q(\gamma)\frac{\partial^\gamma u(x,t)}{\partial |x|^\gamma} d\gamma + f(x,t), \quad (x,t) \in (0,1) \times (0,1], \tag{27}$$

with boundary data

$$u(0,t) = 0, \quad u(1,t) = 0,$$

and initial data

$$u(x,0) = x^2(1-x)^2.$$

Here,

$$P(\alpha) = -2\Gamma(5-\alpha)\cos(\frac{\pi\alpha}{2}), \quad Q(\gamma) = \begin{cases} 0, & 0 < \gamma \le \frac{1}{2}, \\ 2\Gamma(5-\gamma)\cos(\frac{\pi\gamma}{2}), & \frac{1}{2} < \gamma < 1. \end{cases}$$

$$f(x,t) = e^t x^2(1-x)^2 - \int_1^2 P(\alpha)\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} d\alpha - \int_0^1 Q(\gamma)\frac{\partial^\gamma u(x,t)}{\partial |x|^\gamma} d\gamma$$
$$= e^t x^2(1-x)^2 - e^t[R_1(x) + R_1(1-x) - R_2(x) - R_2(1-x)].$$

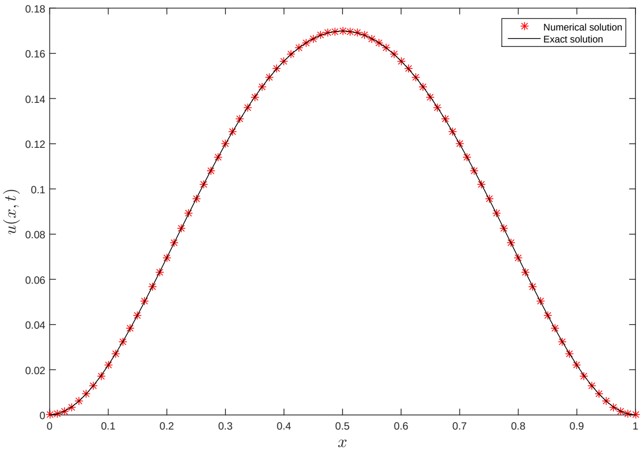

Figure 1: Exact solution and numerical solution with $\sigma = \varrho = \tau = h = 1/80$ at $t = 1.0$.

$$\begin{aligned}
\text{Where, } R_1(x) &= \Gamma(5)\cdot\frac{1}{\ln x}(x^3 - x^2) - 2\Gamma(4)[\frac{1}{\ln x}(3x^2 - 2x) - \frac{1}{(\ln x)^2}(x^2 - x)] \\
&+ \Gamma(3)\frac{1}{\ln x}[6x - 2 - \frac{5x}{\ln x} + \frac{3}{\ln x} + \frac{2x}{(\ln x)^2} - \frac{2}{(\ln x)^2}], \\
R_2(x) &= \Gamma(5)\frac{1}{\ln x}(x^{\frac{7}{2}} - x^3) - 2\Gamma(4)[\frac{1}{\ln x}(\frac{7}{2}x^{\frac{5}{2}} - 3x^2) - \frac{1}{(\ln x)^2}(x^{\frac{5}{2}} - x^2)] \\
&+ \frac{\Gamma(3)}{\ln x}[\frac{35}{4}x^{\frac{3}{2}} - 6x - \frac{1}{\ln x}(6x^{\frac{3}{2}} - 5x - \frac{2x^{\frac{3}{2}}}{\ln x} + \frac{2x}{\ln x})].
\end{aligned}$$

The exact solution of Eq.(27) is $u(x,t) = e^t x^2(1-x)^2$.

Figure 1 exhibits a comparison of the exact and numerical solutions for this example. We can see that the numerical solution is in excellent agreement with the analytical solution. Table 1 shows the error and convergence orders for our method with respect to τ and h. For different σ, ϱ ($\sigma = \varrho = 1/40$ and $\sigma = \varrho = 1/80$), with decreasing $\tau = h$, the convergence orders of τ and h reach second order. Table 2 shows the error and convergence orders with respect to σ and ϱ. For different τ and h ($\tau = h = 1/100$ and $\tau = h = 1/200$), with decreasing $\sigma = \varrho$, we can obtain that the convergence orders of σ and ϱ are also second order. According to the errors and convergence rates in the first two tables, the finite volume method for the Riesz-space distributed-order equations is effective and stable as expected. In Table 3, we present a comparison of the errors and convergence orders between our finite volume method and finite difference method in [47]. Compared to [47], our numerical error is much smaller. Therefore, our method is more effective for the one-dimensional case. However, when comes to high dimensional problem, [47] can apply alternating direction method to reduce the CPU computation time, which is their advantage.

Table 1: The errors and convergence orders with respect to τ and h.

	$\sigma = \varrho = 1/40$		$\sigma = \varrho = 1/80$	
$\tau = h$	error	order	error	order
1/8	2.9352E-03	-	2.9370E-03	-
1/16	7.5119E-04	1.97	7.5301E-04	1.96
1/32	1.8832E-04	2.00	1.9014E-04	1.99
1/64	4.5576E-05	2.05	4.7381E-05	2.00
1/128	9.6818E-06	2.23	1.1439E-05	2.05

4.2. Example 2

Next, we consider the following distributed-order equation [9, 42]:

$$\frac{\partial u(x,t)}{\partial t} = \int_0^2 P(\alpha)\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha}d\alpha. \tag{28}$$

Table 2: The errors and convergence orders with respect to σ and ϱ.

	$\tau = h = 1/200$		$\tau = h = 1/400$	
$\sigma = \varrho$	error	order	error	order
1/2	9.9864E-04	-	1.0031E-03	-
1/4	2.4910E-04	2.00	2.5273E-04	1.99
1/8	5.9049E-05	2.08	6.2526E-05	2.02
1/16	1.1457E-05	2.37	1.4798E-05	2.08
1/32	1.8889E-06	2.60	2.8853E-06	2.36

Table 3: The errors and convergence orders comparison of FVM and FDM in [47] for $\sigma = \varrho = 1/1000$ at $t = 1$.

	FVM		FDM	
$\tau = h$	error	order	error	order
1/20	4.8471E-04	-	3.592E-02	-
1/40	1.2238E-04	1.986	8.916E-03	2.010
1/80	3.0745E-05	1.993	2.204E-03	2.016
1/160	7.7034E-06	1.997	5.388E-04	2.032

According to [9, 42], if $u(x, 0) = \delta(x)$ and $P(\alpha) = Kl^{\alpha-2}[A_1\delta(\alpha - \alpha_1) + A_2\delta(\alpha - \alpha_2)]$, then the solution of Eq.(28) is a convolution of two stable PDFs,

$$u(x,t) = a_1^{-\frac{1}{\alpha_1}} a_2^{-\frac{1}{\alpha_2}} t^{-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}} \int_{-\infty}^{+\infty} L_{\alpha_1,1}\Big(\frac{x-x'}{(a_1 t)^{\frac{1}{\alpha_1}}}\Big) L_{\alpha_2,1}\Big(\frac{x'}{(a_2 t)^{\frac{1}{\alpha_2}}}\Big) dx', \tag{29}$$

where $a_1 = A_1K/l^{2-\alpha_1}$, $a_2 = A_2K/l^{2-\alpha_2}$ and $L_{\alpha,1}$ is the PDF of a symmetric Lévy stable law given by its characteristic function

$$\hat{L}_{\alpha,1}(\xi) = \exp(-|\xi|^{\alpha}).$$

As the exact solution involves convolution and inverse Fourier transform, it is challenging to observe the behavior of $u(x, t)$ directly from Eq.(29). Therefore the numerical solution becomes a promising tool. Without loss of generality, here we consider the numerical solution of Eq.(28) with initial condition $u(x, 0) = \delta(x - 0.5)$, $x \in (0, 1)$ and $K = 2$, $l = 3$, $A_1 = A_2 = 1$. To give the error estimate, here we choose the numerical solution $u(x, t_n) = \sum_{i=1}^{m-1} u_i^n \phi_i(x)$ on a fine grid ($h = 1/500$) as the exact solution. Then we adopt a set of points to calculate the discrete L_2 error on the coarse grids, which is given in the Table 4. We can see that the second order is attained, which shows the stability and reliability of our method again. Now we observe the diffusion behaviour of $u(x, t)$. Figure 3 displays a diffusion behaviour of $u(x, t)$ at different times $t = 1, 5, 10, 20$ that decays with increasing time. Figure 4 and Figure 5 illustrate the impact of α_1 and α_2 on the diffusion behaviour of $u(x, t)$. We can observe that with increasing α_1 or α_2, $u(x, t)$ decays and the effect of α_2 on the diffusion is more significant. Although we give the numerical scheme of Eq.(28), how to apply the scheme to solve the actually problem and establish the connection between the kinetics equation and multifractality need further investigation.

Table 4: The errors and the convergence orders with $\sigma = \varrho = 1/100$, $\alpha_1 = 0.955$, $\alpha_2 = 1.255$ at $t = 1$.

$\tau = h$	error	order
1/160	1.0414E-02	-
1/200	7.3644E-03	1.55
1/320	2.7608E-03	2.09

4.3. *Example 3*

Finally, we consider the following distributed-order equation:

$$\frac{\partial u(x,t)}{\partial t} = \int_1^2 P(\alpha)\frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} d\alpha + \int_0^1 Q(\gamma)\frac{\partial^{\gamma} u(x,t)}{\partial |x|^{\gamma}} d\gamma, \quad (x,t) \in (0,1)\times(0,1], \tag{30}$$

with boundary data

$$u(0,t) = 0, \quad u(1,t) = 0,$$

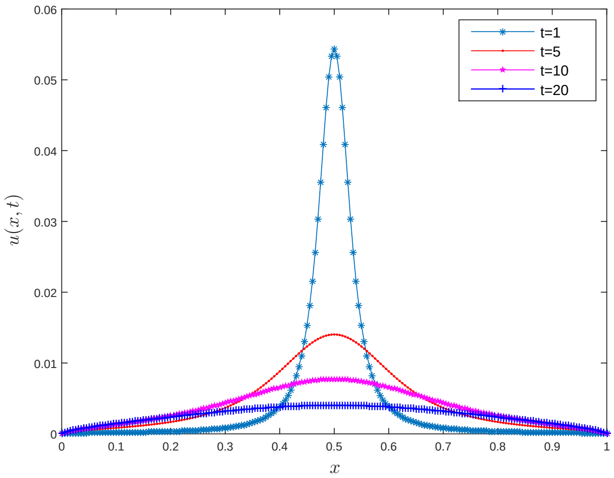

Figure 2: Numerical solution profile of $u(x,t)$ at different t with $\sigma = \varrho = 1/100$, $\tau = h = 1/200$, $\alpha_1 = 0.955$, $\alpha_2 = 1.255$.

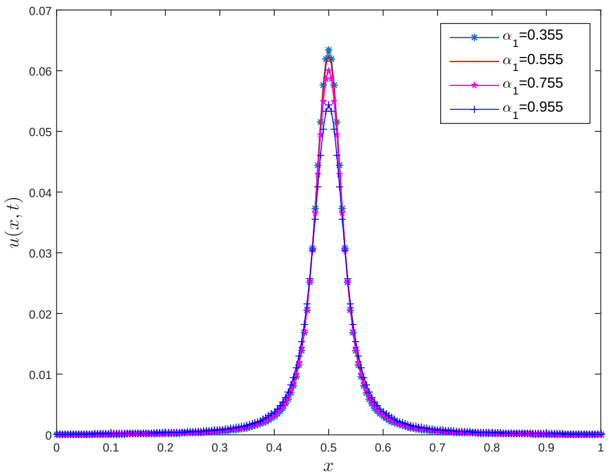

Figure 3: Numerical solution profile of $u(x,t)$ for different α_1 with $\sigma = \varrho = 1/100$, $\tau = h = 1/200$, $\alpha_2 = 1.255$ at $t = 1$.

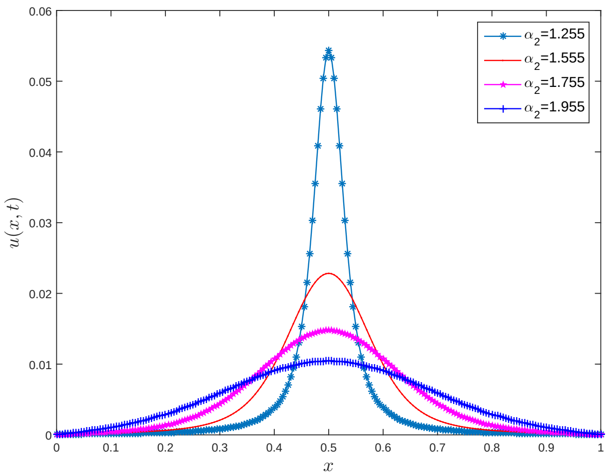

Figure 4: Numerical solution profile of $u(x,t)$ for different α_2 with $\sigma = \varrho = 1/100$, $\tau = h = 1/200$, $\alpha_1 = 0.955$ at $t = 1$.

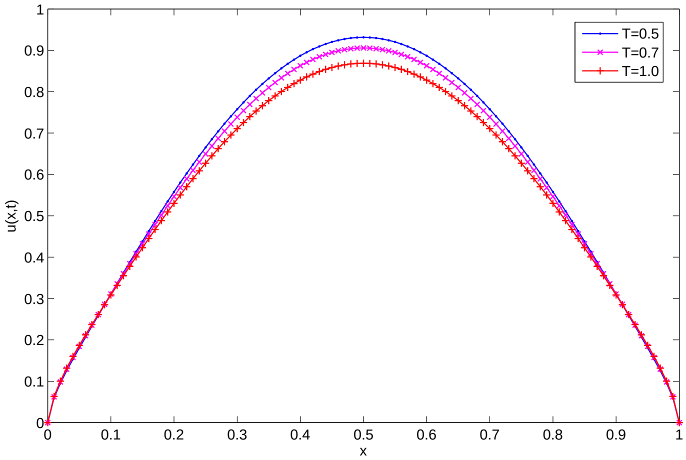

Figure 5: Numerical solution at different times.

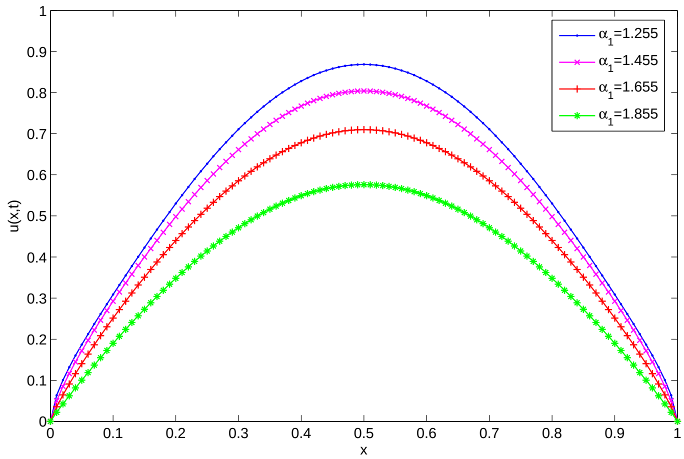

Figure 6: Numerical solution for different α_1.

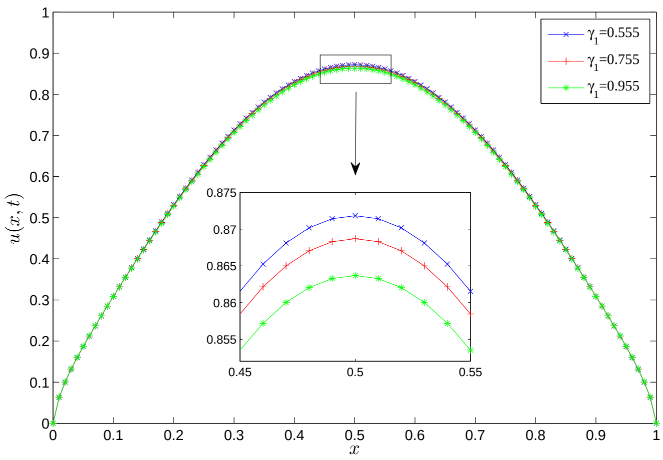

Figure 7: Numerical solution for different γ_1.

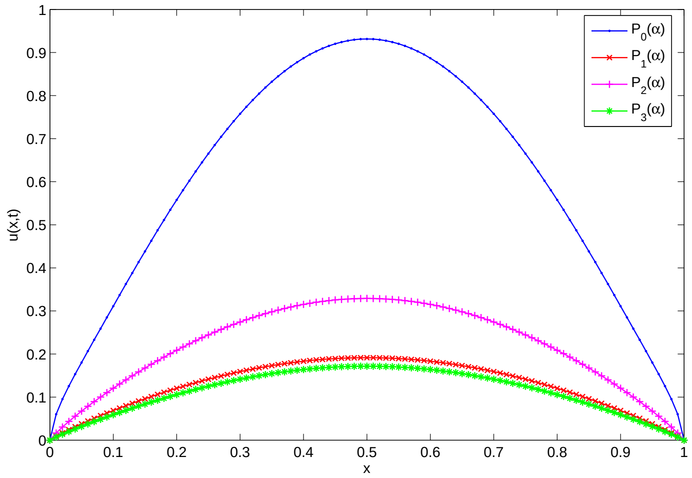

Figure 8: Numerical solution for different $P(\alpha)$.

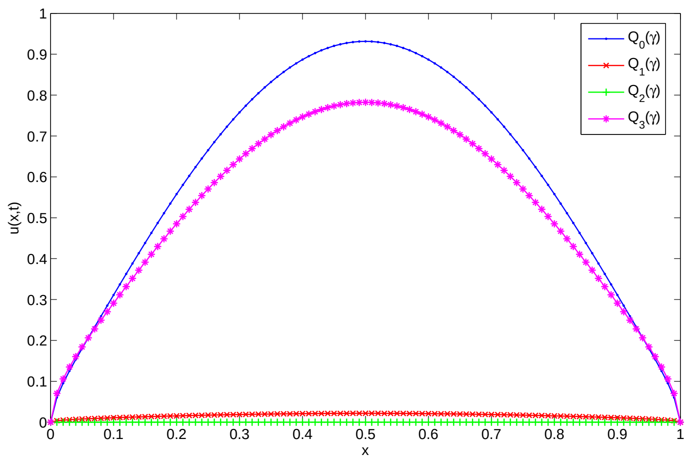

Figure 9: Numerical solution for different $Q(\gamma)$.

and initial data

$$u(x,0) = \sin(\pi x).$$

Here we choose

$$P(\alpha) = l^{\alpha-2} K A_1 \delta(\alpha - \alpha_1), \quad Q(\gamma) = l^{\gamma-2} K A_2 \delta(\gamma - \gamma_1),$$

$\sigma = \varrho = \tau = h = 1/100$, $l = 3$, $K = 1$, $A_1 = 8$ and $A_2 = 2$. Firstly, in Figure 5 we exhibit the diffusion behavior of $u(x,t)$ at the different times $T = 0.5$, 0.7, 1.0 with $\alpha_1 = 1.255$, $\gamma_1 = 0.755$, which decays with increasing time. Then we consider the diffusion behavior of $u(x,t)$ by choosing distinct α_1 at $T = 1.0$ with $\gamma_1 = 0.755$, which is shown in Figure 6. Similarly, in Figure 7, we observe the diffusion behavior of $u(x,t)$ by adopting distinct γ_1 at $T = 1.0$ and zoom the $[0.45, 0.55] \times [0.852, 0.875]$ part. We can similarly see that with increasing α_1 or γ_1, $u(x,t)$ decays and the effect of α_1 on the diffusion is more significant. Finally, we exhibit a comparison graphics of the diffusive behavior of $u(x,t)$ by selecting different $P(\alpha)$ or $Q(\gamma)$ at $T = 0.5$ in Figure 8 and Figure 9 at time $T = 0.5$, respectively, in which $P_0(\alpha) = 8 \cdot 3^{\alpha-2}\delta(\alpha - 1.255)$, $P_1(\alpha) = \frac{1}{\alpha}$, $P_2(\alpha) = \frac{1}{\alpha^2}$, $P_3(\alpha) = 3^{\alpha-2}$ and $Q(\gamma) = 2 \cdot 3^{\gamma-2}\delta(\gamma - 0.755)$ in Figure 8, while $Q_0(\gamma) = 2 \cdot 3^{\gamma-2}\delta(\gamma - 0.755)$, $Q_1(\gamma) = \frac{1}{\gamma}$, $Q_2(\gamma) = \frac{1}{\gamma^2}$, $Q_3(\gamma) = 3^{\gamma-2}$ and $P(\alpha) = 8 \cdot 3^{\alpha-2}\delta(\alpha - 1.255)$ in Figure 9. We can conclude that the both $P(\alpha)$ and $Q(\gamma)$ have effects on the diffusion behavior of $u(x,t)$.

5. Conclusions

In this paper, we have investigated a second order in both space and time numerical scheme for the Riesz space distributed-order advection-diffusion equation. We prove that the Crank-Nicolson scheme is unconditionally stable and convergent with second order accuracy $O(\sigma^2 + \varrho^2 + \tau^2 + h^2)$. Three numerical examples are presented to show the effectiveness of our computational method. In the future, we would like to develop the finite volume method to solve time distributed-order, time-space distributed-order advection-diffusion equations in one-dimensional and two-dimensional space. Moreover, we shall consider other computational methods to improve the convergence rate.

Acknowledgements

Author Turner wishes to acknowledge that this research was partially supported by the Australian Research Council (ARC) via the Discovery Project DP150103675, and author Li wishes to acknowledge the National Natural Science Foundation of China grant 11301040, 11226166 and the State Scholarship Fund from China Scholarship Council.

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